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Perez, Jérome; Füzfa, André; Carletti, Timoteo; Melot, Laurence; Guedezounme, Sêcloka Lazare

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RESEARCH ARTICLE

The Jungle Universe: coupled cosmological models in a Lotka–Volterra framework

Jérôme Perez · André Füzfa · Timoteo Carletti · Laurence Mélot · Lazare Guedezounme

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Abstract In this paper, we exploit the fact that the dynamics of homogeneous and isotropic Friedmann–Lemaître universes is a special case of generalized Lotka–Volterra system where the competitive species are the barotropic fluids filling the Universe. Without coupling between those fluids, Lotka–Volterra formulation offers a pedagogical and simple way to interpret usual Friedmann–Lemaître cosmological dynamics. A natural and physical coupling between cosmological fluids is proposed which preserves the structure of the dynamical equations. Using the standard tools of Lotka–Volterra dynamics, we obtain the general Lyapunov function of the system when one of the fluids is coupled to dark energy. This provides in a rigorous form a generic asymptotic behavior for cosmic expansion in presence of coupled species, beyond the standard de Sitter, Einstein-de Sitter and Milne cosmologies. Finally, we conjecture that chaos can appear for at least four interacting fluids.

Keywords Cosmology · Coupled models · Dynamical systems · Dark energy · Predator-prey system

J. Perez
ENSTA-Paristech Applied Mathematics Laboratory 828, Boulevard des Marechaux, 91762 Palaiseau Cedex, France

A. Füzfa (✉) · T. Carletti
Namur Center for Complex Systems (naXys), University of Namur, Namur, Belgium
e-mail: andre.fuzfa@unamur.be

L. Mélot
Faculty of Computer Science, University of Namur, Namur, Belgium

L. Guedezounme
UNESCO Chair of Mathematical Physics and Applications, University of Abomey-Calavi, Abomey-Calavi, Benin

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1 Introduction

"*Modern dynamical system theory can help us in understanding the evolution of cosmological models*". This remark in the introduction of the classical Wainwright and Ellis textbook [1], is both true and full of sense: it is often confronting approaches that one can really understand problems. In the context of spatially homogeneous and anisotropic universes it was fully investigated since the pioneering famous works of the 70s (e.g. [2] for the BKL conjecture [3], for mixmaster universes or [4] for the triangle map) up to mathematical proofs and deep understanding of this dynamics by Ringström [5] or extension to space–time dimensions $D > 4$ e.g. [6].

The situation is a little bit less rich in the context of spatially homogeneous and isotropic universes. Concerning the classical Friedmann–Lemaître (FL) Universes containing non-interacting barotropic fluids, various scenari for the fate of the Universe have been popularized. Among these are the Big Chill—when cosmic expansion is endless or the Big Crunch—a final singularity of same nature than the Big Bang for spatially closed cosmologies with vanishing or small cosmological constant, or more recently Big Rip [7], when Universe's scale factor become infinite at a finite time in the future. Dynamical systems tools have allowed some important results in the question of future asymptotic behavior of cosmic expansion, for instance by demonstrating the existence of attracting regimes and scaling solutions in quintessence models [8–11]. Solutions to cosmological dynamics consist of time evolution of density parameters associated to the barotropic fluids usually invoked to model matter contents of the universe. The fate of the Universe is completely related to its matter content. For example, Big Rip singularity occurs when Universe contains the so-called "Phantom dark energy" associated to a barotropic fluid with equation of state $p = \omega\rho$ where the barotropic index $\omega < -1$. Recently, it appears that cosmological FL models with *interacting* components have gained interest because it might be expected that the most abundant components in the present Universe, dark energy (DE) and dark matter (DM), probably interact with each other. Such interactions are considered by some authors to be promising mechanisms to solve some of the ΛCDM problems like coincidence (see for instance [12–15] and references therein). In the literature the coupling between interacting fluids is generally time-independent and quadratic in energy density $\mathcal{Q}_{ij} = \gamma\rho_i\rho_j$ (see [16] and reference therein) where γ is a dimensioned constant or time-dependant and polynomial in energy density $\mathcal{Q}_{ij} = \gamma H\rho_i^m\rho_j^{m-n}$ where H is the Hubble parameter and m and n relevant integers (see [17] and references therein). In these two categories of papers, new behaviors are speculated for cosmological dynamics with non-linear interactions. In particular the existence of cycles have been postulated if one of the species is barotropic with an index $\omega < -1$.

In this paper, we present for the first time FL Universe containing fluids in interaction as a particular case of the well known Lotka–Volterra system. This formulation is possible when one considers the system in term of density contrasts $\Omega_i = \frac{8\pi G}{3H^2}\rho_i$ evolving through the variable $\ln a$ where a is the scale factor of the FL universe. On the one hand this make us able to use a lot of standard techniques of dynamical systems analysis in the context of cosmology; and, on the other hand, this allows (as billiards do for anisotropic models) a global comprehension of this important cosmological problem.

When there are no interactions between constitutive fluids, this formulation allows to interpret those dynamics in a pedagogical way through an intuitive and simple formulation. The FL cosmological dynamics can then be seen as a competition between several species, each associated to one of the fluids filling the universe. Those species all compete for feeding upon the same resource which is spatial curvature. The usual asymptotic states of FL dynamics, de Sitter, Einstein-de Sitter and Milne universes, can all be seen as a particular equilibrium between cosmic species. This is the simplest picture of the Jungle Universe.

In this paper we propose a new kind of coupling between fluids of the form $\mathcal{Q}_{ij} = \gamma H^{-1}\rho_i\rho_j$. This is not an ad hoc approach since this is the only way to preserve the natural Lotka–Volterra form of the FL dynamics. Moreover this time-dependent coupling, once analyzed following the scaling cosmology method presented by Zimdahl and Pavon [18], shows that it grows with the cosmic time. This last property makes this ansatz relevant both with observational constraints and to avoid the coincidence problem.

The paper is structured as follow: in Sect. 2 we show that FL cosmological dynamics is actually a generalized Lotka–Volterra system; in Sect. 3 we interpret the FL cosmological dynamics in terms of the generalized Lotka–Volterra system: The Jungle Universe; in Sect. 4, we show how a general and physical interaction between two fluids can preserve the structure of this dynamical system and we obtain the general Lyapunov function for this kind of dynamics; in Sect. 5, we generalize the formulation to N directly coupled species and focus in particular to triads ($N = 3$) and quartets ($N = 4$); finally, we draw some conclusions in Sect. 6.

Notation In what follows, vectors are written bold faced (e.g. $\mathbf{r} \in \mathbb{R}^n$) and the associated coordinates in the canonical basis are denoted by the italic corresponding letters with an index (e.g. $\mathbf{r} = (r_1, \ldots, r_n)^\top$).

2 Friedmann–Lemaître cosmology as generalized Lotka–Volterra dynamical systems

Taking into account a cosmological constant Λ, Einstein's equations of general relativity write

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \chi T_{\mu\nu}$$

where $g_{\mu\nu}$ and $R_{\mu\nu}$ are respectively the metric and the Ricci tensors, R is the scalar curvature (contraction of the Ricci), $T_{\mu\nu}$ is the stress-energy tensor and $\chi = 8\pi G c^{-4}$. The general paradigm of standard cosmology consists of imposing Friedmann–Lemaître–Robertson–Walker metric as an isotropic and homogeneous description of the universe i.e.

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

where $a(t)$ and k are respectively the scale factor and the curvature parameter, t and (r, θ, ϕ) being the synchronous time and usual spherical coordinates, respectively. If one assumes that this universe is filled by a perfect fluid of density ρ, pressure p and quadri-velocity field u_μ for which$T_{\mu\nu} = \left(\rho + c^{-2} p\right) u_\mu u_\nu - p g_{\mu\nu}$, it is well known that the dynamics of the universe are governed by Friedmann–Lemaître and conservation equations:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{3} \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{2}$$

$$\dot{\rho} = -3H\left(\rho + \frac{p}{c^2}\right) \tag{3}$$

where $H(t) = \dfrac{\dot{a}}{a}$ is the Hubble parameter and a dot over a quantity indicates a derivation with respect to the synchronous time t, the independent variable of the cosmological differential system. Both parameters k and Λ might be seen as fixing the spatial and intrinsic curvature of the geometry.[1] Among the three above equations, only two are independent since all are related through the second Bianchi identities. The remaining two equations still include three unknown functions: $\rho\,(t)\,,\ p\,(t)$ and $a\,(t)$. This under-determination can be raised by introducing an equation of state for the matter fluids. For example, barotropic fluids are such that $p = \omega\rho$ where the constant ω is called the barotropic index. In a general physical way, this index ranges from $\omega_{\min} = -1$ for scalar field frozen in unstable vacuum to $\omega_{\max} = +1$ for stiff matter (e.g. free scalar field) where sound velocity equals to speed of light. In this paper we generally restrict our analysis to such barotropic fluids in general relativity, values of $\omega \notin [-1, 1]$ generally correspond to other theories of gravity.

Following standard procedure, we rewrite the above equations in terms of density parameters for matter $\Omega_m = \frac{8\pi G\rho}{3H^2}$, cosmological constant $\Omega_\Lambda = \frac{\Lambda}{3H^2}$, curvature $\Omega_k = -\frac{k}{3a^2H^2}$ and deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$. Friedmann–Lemaître equations and energy conservation write for barotropic fluids therefore become

$$\begin{cases} 1 = \Omega_m + \Omega_\Lambda + \Omega_k \\ q = \frac{1}{2}\Omega_m\,(1 + 3\omega) - \Omega_\Lambda \\ \dot{\rho} = -3H\rho\,(1 + \omega) \end{cases}$$

Please note that the latter equation can be directly integrated for constant equation of state to give $\rho \sim a^{-3(1+\omega)}$.

Finally, we rewrite the above equations by changing the independent variable to the number of efoldinds $\lambda = \log(a)$ and noting $'$ for λ-derivatives, one gets

$$\begin{cases} 1 = \Omega_m + \Omega_\Lambda + \Omega_k \\ \Omega_m' = \Omega_m\left[-\,(1 + 3\omega) + (1 + 3\omega)\,\Omega_m - 2\Omega_\Lambda\right] \\ \Omega_\Lambda' = \Omega_\Lambda\left[2 + (1 + 3\omega)\,\Omega_m - 2\Omega_\Lambda\right] \end{cases}$$

[1] If one interprets the cosmological constant as the curvature associated to vacuum.

The dynamics of the Friedmann–Lemaître universe is contained in the two last equations which form a differential system of generalized Lotka–Volterra [19–21] equations well known in population dynamics (see "Appendix 1" for an introduction). As a matter or fact, introducing the dynamical vector $\mathbf{x} = (\Omega_m, \Omega_\Lambda)^\top$and the capacity vector $\mathbf{r} = (-(1+3\omega), 2)^\top$, for $i = 1, 2$ we have $x_i' = x_i\ f_i(\mathbf{x})$ where the vector function $\mathbf{f}(\mathbf{x}) = \mathbf{r} + A\mathbf{x}$ is linear in the variables x_i, the community matrix A being defined by

$$A = \begin{bmatrix} (1+3\omega) & -2 \\ (1+3\omega) & -2 \end{bmatrix}$$

This formulation allows us to assimilate the dynamics of Friedmann–Lemaître universes to those of a competition between species, represented by Ω_m and Ω_Λ, for the resources in Ω_k. This point of view is not anecdotal and will reveal a lot of benefit: such equations are very well known to the dynamical system specialist, it allows a lot of intuitive non trivial results, establish an analogy that will help us deriving new cosmological behavior for coupled models besides of providing a pedagogic and interesting insight on cosmic expansion.

First of all, it is easy to see that orbits cannot cross the $\Omega_m = 0$ or $\Omega_\Lambda = 0$ axes which are orbits themselves.[2] As the matrix A fully degenerates (rank equal to 1) it is clearly not invertible, equilibrium points must lie on axis. In particular as denoted by Uzan and Lehoucq [22] or Hobson et al. [23] using a slightly different dynamical system, there exists 3 equilibria which are Milne universe $\mathbf{x}_0 = (0, 0)$, Einstein-de Sitter universe $\mathbf{x}_1 = (1, 0)$ and de Sitter universe $\mathbf{x}_2 = (0, 1)$. Using the large knowledge of such systems from bio-mathematics (e.g. [24,25]) the $\mathbf{r}$ vector contains the intrinsic birth or death rates of the species. The dynamics of competitive Lotka–Volterra systems with such a degenerate matrix is well known:

- If the initial condition is located in the positive quadrant $Q^+ = \{\Omega_m > 0\} \times \{\Omega_\Lambda > 0\}$ then $\mathbf{x} \to \mathbf{x}_2$ when t or λ goes to infinity, the reason of this attractive character of the de Sitter universe is uniquely contained in the fact that $r_2 \geq r_1$ for all physical values of the barotropic index ω. If we extend values of ω considering phantom dark energy instead of pressureless matter by letting $\omega < -1$ the attractor become the (phantom DE-dominated) Einstein-de Sitter universe ($\mathbf{x}_1$) simply because in this case $r_1 \geq r_2$. This is obvious since in this case the energy density of the phantom DE grows like a power-law with the scale factor ($\rho_{DE} \sim a^{-3(1+\omega)}$ where $\omega < -1$), therefore asymptotically dominating the constant density associated to the cosmological term.
- If the initial condition lies on the Ω_m axis the attractor is the Einstein-de Sitter universe if $\omega < -\frac{1}{3}$ and Milne universe ($\mathbf{x}_0$) if $\omega \geq -\frac{1}{3}$. This is obvious since, in the absence of a cosmological constant ($\Omega_\Lambda = 0$), the competition is left between matter and curvature energy densities, the latter decreasing as a^{-2}. Therefore, asymptotic dominance of matter is only possible when $\omega < -1/3$, so that the related density can eventually dominate (since it scales as $\rho_m \sim a^{-3(1+\omega)}$).

[2] This point doesn't exclude the possibility of negative values for Λ in the Friedmann–Lemaître equations, but it doesn't allow changes for Λ's signum in a given Friedmann–Lemaître universe.

- If the initial condition lies on the Ω_Λ axis the attractor is the de Sitter universe for any values of ω. Once again, this is obvious since asymptotically the constant energy density of the cosmological term will dominate the decreasing energy density related to the curvature.

These results are well known and presented in a slightly different manner in [22] or [23]. The new point here is the dynamical population formulation of the problem and interesting results will be derived through usual techniques in dynamical system theory. We will also present new cosmological consequences on coupled models which are directly inspired by the analogy with evolution of populations in competition. One possibility consists of investigating how far the natural cyclic orbits appearing usually in population dynamics could appear in standard cosmology. This is the object of the next section.

3 Multi-components Friedman–Lemaître universes: Jungle Universes

In the latter section we have presented the generalized Lokta–Volterra formulation for the dynamics of usual Friedmann–Lemaître universe with non-vanishing cosmological constant. In particular we have only considered one simple barotropic fluid characterized by a given value of ω. We can generalize this situation to the more complicated yet realistic case where the universe is filled by several kinds of barotropic fluids without any direct interactions. In this section, we consider for example baryonic matter (b—indiced and for which $\omega_\mathrm{b} = 0$) and radiation (r—indiced and for which $\omega_\mathrm{r} = \frac{1}{3}$). It is well known that the repulsive feature obtained with a positive cosmological constant can also advantageously be obtained through some dark energy fluid component (e—indiced) associated to a barotropic index $\omega_\mathrm{e} \in [-1, -1/3]$; the cosmological constant term could then be obtained taking $\omega_\mathrm{e} = -1$. In the following, roman indexes refer to the fluid component considered.

The cosmological term in Friedmann–Lemaître equations can therefore be removed, introducing the densities $\Omega_x = \frac{8\pi G\rho_x}{3H^2}$ for $x = \mathrm{b}, \mathrm{r}$ and e including the conservation of each kind of fluids they write

$$\begin{aligned}
1 &= \Omega_\mathrm{b} + \Omega_\mathrm{r} + \Omega_\mathrm{e} + \Omega_\mathrm{k} \\
2q &= \Omega_\mathrm{b} + 2\Omega_\mathrm{r} + (1 + 3\omega_\mathrm{e})\,\Omega_\mathrm{e} \\
(\ln \rho_x)' &= -3(1 + \omega_x) \quad \text{for } x = \mathrm{b}, \mathrm{r} \text{ and e;}
\end{aligned}$$

A basic calculus shows that $(\ln H)' = -q - 1$ hence Friedmann–Lemaître equations write

$$\frac{\Omega_x'}{\Omega_x} = (\ln \Omega_x)' = \Omega_\mathrm{b} + 2\Omega_\mathrm{r} + (1 + 3\omega_\mathrm{e})\,\Omega_\mathrm{e} - 3\omega_x - 1 \quad \text{for } x = \mathrm{b}, \mathrm{r} \text{ and e}$$

The three dimensional differential system for Ω_e, Ω_b and Ω_r is always a generalized Lotka–Volterra form with a fully degenerate community matrix. The dynamics is then always governed by the capacity vector $\mathbf{r} = [-1, -2, -3\omega_\mathrm{e} - 1]$ which actually rules the asymptotic behavior. Besides of the origin, there is now one additional equilibrium on each axis and if $\mathbf{r}$ possesses a component which is greater than all others, the corresponding equilibrium with this component maximal is globally stable over the positive orthant. This smart result is sufficient to claim that dark energy (for which $\omega_e \in [-1, -1/3]$) corresponds to this $\mathbf{r}$ maximal components and then the universe such that $\Omega_b = \Omega_r = 0$ and $\Omega_e = 1$ is globally stable out from axis $\Omega_b = 0$ and $\Omega_r = 0$.

This three dimensional situation is readily generalizable to any number of non interacting fluids each governed by a separated conservation equation. The dynamical behavior is asymptotically always the same: the system evolves like a competitive one in which all species (predators) are fed by the same prey (which is curvature…). Asymptotically and out of axis, only one species survives, the one which possesses the greater value of $-3\omega_x - 1$. This species is always the dark energy fluid in our physical hypotheses $\omega \in [-1, 1]$. Once the Universe is filled with even a small amount of dark energy, there is no way it cannot dominate forever the fate of the cosmos. This is Jungle Law for a Jungle Universe. Fortunately, this will cease to be true, as we shall see in the next section, if dark energy is not so dark, but it is in interaction with the other components.

4 Cooperation in the Jungle Universes

4.1 General dynamics with dark coupling

In the last sections we have presented a way to express the dynamics of Friedmann–Lemaître universes using generalized Lotka–Volterra differential system theory. This also offers new perspectives in determining cosmological analogues of specific cases in competitive dynamics. It is well known that the generic dynamics of such systems contains limit cycles or periodic orbits. We will describe in this section how direct coupling can be used to bring such a behavior in the context of cosmology.

When the fluids filling the universe are not interacting with each other, the community matrix of the generalized Lotka–Volterra system must have the same rows and then must be fully degenerated. In order to make its rank greater than one, we must introduce coupling between species, i.e. interactions between cosmological fluids. On the other hand, this kind of interactions is broadly used in cosmology, with the coupling between inflaton and radiation during reheating (e.g. [26]) or the one between dark matter and dark energy (e.g. [8,27–29]), or even the decay of heavy matter particles like WIMPS into light relativistic particles (e.g. [30]). Modern cosmology make strong use of coupled fluids for a variety of purposes, therefore making this study of coupled models in terms of Lotka–Volterra systems of first heuristic interest.

In order to show the phenomenon we will present in this section the situation where the universe contains radiation, baryonic matter, dark matter (d—indiced),[3] dark energy and we suppose a coupling between the two dark components. This constitutes a coupled quintessence scenario [28,29]. On one hand, it is necessary to preserve the global energy conservation as imposed by Noether theorem and Poincare invariance, energy transfer must compensate in the global energy balance. Hence, at each time,

[3] Although both are pressureless with $\omega = 0$, we split both to allow for different couplings.

the part of the energy taken by the first component must be given to the other to which it couples. To achieve this, conservation equations for two coupled dark fluids must be of the following form:

$$\begin{cases} \dot{\rho}_{\mathrm{d}} = -3H\rho_{\mathrm{d}}\left(1+\omega_{\mathrm{d}}\right) + \mathcal{Q} \\ \dot{\rho}_{\mathrm{e}} = -3H\rho_{\mathrm{e}}\left(1+\omega_{\mathrm{e}}\right) - \mathcal{Q} \end{cases}$$

where $\mathcal{Q}$ represents the energy transfer. This coupling leaves unchanged the global energy-momentum conservation, it is then invisible in standard general relativity and it glimpses at (micro-)physics describing dark components of the universe. In literature, one usually finds that this energy transfer is arbitrarily expressed as a linear combination of the dark sector densities:

$$\mathcal{Q} = A_{\mathrm{d}}\rho_{\mathrm{d}} + A_{e}\rho_{\mathrm{e}}$$

where the coefficients are either proportional to Hubble parameter H either constant (see [8,27–29]). In this paper, we introduce a new non-linear parametrization of the energy transfer that allows us matching the coupled model to a general Lotka–Volterra system. This ansatz is given by

$$\mathcal{Q} = \frac{8\pi G}{3H}\varepsilon\rho_{\mathrm{e}}\rho_{\mathrm{d}} = \varepsilon H\Omega_{\mathrm{e,d}}\rho_{\mathrm{d,e}} \tag{4}$$

where the *coupling parameter* ε *is a positive constant*. This form of coupling is required to preserve the fundamental generalized Lotka–Volterra form of the FL dynamical system, but it also has an interesting phenomenological motivation. When the energy transfer is written $\mathcal{Q} \sim H\rho_{\mathrm{d,e}}$, this means that the energy moved from one species to the other varies with the volume of the cosmological fluid. Our original suggestion, $\mathcal{Q} \sim H\Omega_{\mathrm{e,d}}\rho_{\mathrm{d,e}}$, now assumes that, in addition to the abovementioned volume variation, the energy transfer $\mathcal{Q}$ is also proportional to the *cosmological abundance* of one species. This is very different than assuming that the energy transfer is related to the *local energy density*, as in an expression like $\mathcal{Q} \sim \rho_{\mathrm{e,d}}\rho_{\mathrm{d,e}}$, which could be encountered in the case of two species undergoing reactions at the microscopic level (see [31]). The energy transfer considered here, $\mathcal{Q} \sim H\Omega_{\mathrm{e,d}}\rho_{\mathrm{d,e}}$, is proportional to the *proportion* $\Omega_{\mathrm{e,d}}$ of one species e, d. This glimpses at more complex microscopic interactions between the coupled species, maybe including agglomeration or saturation effects, that should be further investigated.

In conclusion, the motivation for this particular coupling is twofold: a mathematical one, by ensuring a Lotka–Volterra interpretation of cosmological dynamics and a phenomenological one, by supposing the energy transfer is proportional to the cosmological abundance of the coupling species. Such coupling has still to be motivated by microscopic physics (see also [31]), maybe through sophisticated microscopic interactions with saturation effects that affects the amplitude of the energy transfer when one of the coupled species grow.

In addition, if we consider only the dark components of the universe—the dark plane where $\Omega_{\mathrm{b}} = \Omega_{\mathrm{r}} = 0$—in the case where the non baryonic dark matter is non-relativistic and pressureless, i.e. $\omega_{\mathrm{d}} = 0$ and following the scaling method introduced by Zimdahl and Pavon [18], we get

$$\frac{d}{dt}\left(\frac{\rho_{\rm d}}{\rho_{\rm e}}\right)=\frac{\rho_{\rm d}}{\rho_{\rm e}}\left[\frac{\dot\rho_{\rm d}}{\rho_{\rm d}}-\frac{\dot\rho_{\rm e}}{\rho_{\rm e}}\right]=\frac{\rho_{\rm d}}{\rho_{\rm e}}\left[3H\omega_{\rm e}+\frac{8\pi G\varepsilon\left(\rho_{\rm d}+\rho_{\rm e}\right)}{3H}\right]$$

In the dark plane we have

$$\frac{8\pi G}{3}\left(\rho_{\rm d}+\rho_{\rm e}\right)=H^2$$

as $H=\frac{d\ln a}{dt}$ we obtain

$$\frac{d}{dt}\left(\frac{\rho_{\rm d}}{\rho_{\rm e}}\right)=\frac{\rho_{\rm d}}{\rho_{\rm e}}H\left[3\omega_{\rm e}+\varepsilon\right]\implies\frac{d\ln\left(\frac{\rho_{\rm d}}{\rho_{\rm e}}\right)}{dt}=\left[3\omega_{\rm e}+\varepsilon\right]\frac{d\ln a}{dt}$$

and finally

$$\frac{\rho_{\rm d}}{\rho_{\rm e}}=\frac{\rho_{\rm d,0}}{\rho_{\rm e,0}}\left(\frac{a}{a_0}\right)^{3\omega_{\rm e}+\varepsilon}\tag{5}$$

which avoid the coincidence problem if $3\omega_{\rm e}+\varepsilon>0$.

Since the Raychaudhuri equation (2) and consequently $(\ln H)'$ are left unchanged by the introduction of such couplings,[4] but we have now

$$\begin{aligned}(\ln\Omega_{\rm d})'&=(\ln\rho_{\rm d})'+2q+2\\&=\Omega_{\rm b}+(1+3\omega_{\rm d})\,\Omega_{\rm d}+2\Omega_{\rm r}+(\varepsilon+1+3\omega_{\rm e})\,\Omega_{\rm e}-(3\omega_{\rm d}+1)\end{aligned}$$

and

$$\begin{aligned}(\ln\Omega_{\rm e})'&=(\ln\rho_{\rm e})'+2q+2\\&=\Omega_{\rm b}+(1+3\omega_{\rm d}-\varepsilon)\,\Omega_{\rm d}+2\Omega_{\rm r}+(1+3\Omega_{\rm d})\,\Omega_{\rm d}-(3\omega_{\rm e}+1)\,.\end{aligned}$$

The other two remaining equations for $\Omega_{\rm b}$ and $\Omega_{\rm d}$ are not affected by the dark coupling. With this coupling and under these last hypotheses the generalized Lotka–Volterra equations associated to isotropic, homogeneous and barotropic fluid filled universe for the dynamical variable $\mathbf{x}=(\Omega_{\rm b},\Omega_{\rm d},\Omega_{\rm r},\Omega_{\rm e})^\top$ are defined by a capacity vector $\mathbf{r}$ and a community matrix A such that

$$A=\begin{bmatrix}1&1&2&1+3\omega_{\rm e}\\1&1&2&\varepsilon+1+3\omega_{\rm e}\\1&1&2&1+3\omega_{\rm e}\\1&1-\varepsilon&2&1+3\omega_{\rm e}\end{bmatrix}\text{ and }\mathbf{r}=\begin{bmatrix}-1\\-1\\-2\\-1-3\omega_{\rm e}\end{bmatrix}\tag{6}$$

As desired this matrix is not fully degenerate but generally has rank equals to 3. This dynamic is characterized by five equilibria in the positive quadrant which are $\tilde{\mathbf{x}}^0=(0,0,0,0)^\top$, $\tilde{\mathbf{x}}^1=(0,0,1,0)^\top$, $\tilde{\mathbf{x}}^2=(0,0,0,1)^\top$, $\tilde{\mathbf{x}}^3=(1-\alpha,\alpha,0,0)^\top$

[4] This is so since gravity is still minimally coupled to matter fluids.

with $\alpha\in]0,1]$ and $\tilde{\mathbf{x}}^4=\varepsilon^{-1}\left(0,-1-3\omega_e,0,1\right)^\top$ the first four being globally unstable while the last is by far the most interesting.

Provided that

$$\omega_e<-\frac{1}{3}\text{ and }\varepsilon>-3\omega_e>0 \tag{7}$$

the equilibrium $\tilde{\mathbf{x}}^4$ is always in the positive quadrant but it is no more hyperbolic and two complex eigenvalues, namely

$$\lambda_\pm=\pm i\sqrt{\frac{\left|9\omega_e^2+3\varepsilon\omega_e+3\omega_e+\varepsilon\right|}{\varepsilon}},$$

occur in the spectrum of the linearized dynamics around $\tilde{\mathbf{x}}^4$. A precise analysis of the dynamical behavior of the system is then required. In order to do this we have decomposed the job into two parts:

1. In a first step (Sect. 4.2), we have restricted the analysis to the dark plane (Ω_d,Ω_e) where we have rigorously proven that the dynamic is generally cyclic; this proof was exhibited using a general new Lyapunov function.
2. In a second step (Sect. 4.3), we have shown that this dark plane is attractive for all orbits whose initial conditions belong to the hyper-tetrahedron

$$T_4=\{\Omega_b>0\}\cup\{\Omega_d>0\}\cup\{\Omega_r>0\}\cup\{\Omega_e>0\}\cup\{\Omega_b+\Omega_d+\Omega_r+\Omega_e<1\}. \tag{8}$$

4.2 Cyclicity of orbits in the dark plane: Lyapunov function for FL dynamics

Cyclic behaviors for coupled FL cosmologies has been proposed by some papers (see [16] or [17]) but in each cases the polytropic index ω_e required for dark energy is less than -1. Moreover their conclusion of cyclic orbits are obtained from linear analysis around non hyperbolic equilibria; however, it is well known in dynamical system analysis that one cannot conclude anything in this case: we have devoted the "Appendix 2" to this point. In order to be sure to have such cyclic behavior it is necessary in this context to use more refined tools like Lyapunov functions.

Such tools are fundamental in physics but generally it is very hard to find them. However, it is possible in a very general manner in the context of our coupled FL dynamics thanks to the fact that it is a generalized Lotka–Volterra system.

In a pedagogical objective we propose to show how to construct such kind of fundamental functions for Jungle Universes containing a coupling in the dark sector. Using this method, this result is widely generalizable to another cases.

In the so-called dark-plane ($\Omega_b=\Omega_r=0$) and with the notations $x=\Omega_d$ and $y=\Omega_e$, the dynamics is then governed by the generalized Lotka–Volterra system

$$\begin{cases}x'=x\left[x+(1+3\omega_e+\varepsilon)\,y-1\right]\\ y'=y\left[(1-\varepsilon)\,x-(1+3\omega_e)\,y-1-3\omega_e\right]\end{cases}$$

hence

$$\mathbf{r}_2 = \begin{bmatrix} -1 \\ -1-3\omega_\mathrm{e} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1+3\omega_\mathrm{e}+\varepsilon \\ 1-\varepsilon & 1+3\omega_\mathrm{e} \end{bmatrix}$$

As indicated in the previous section, for the dark sector and under the restriction (7), there is a unique equilibrium in the strict positive quadrant, namely $(\tilde{x}, \tilde{y}) = \varepsilon^{-1}(-1-3\omega_\mathrm{e}, 1)$. The interval of study $\omega_\mathrm{e} \in \left(-\infty, -\frac{1}{3}\right]$ include the cosmological constant ($\omega_\mathrm{e} = -1$), a large variety of quintessence scenario, and of course the hypothetical phantom dark matter such that $\omega_\mathrm{e} < -1$. In all these cases, non hyperbolic eigenvalues appear for a sufficient strength of coupling $\varepsilon > |3\omega_\mathrm{e}|$, for lighter coupling $\tilde{\mathbf{x}}^4$ is unstable.

Let us turn now to the construction of the Lyapunov function.

Using a bit of intuition and dynamical population analysis tools (e.g. [24]) one can use the function $V_{\varepsilon,\omega_\mathrm{e}}(x, y) = x^\alpha y^\beta (a + bx + cy)$ where α and β are functions of ε and ω_e; a, b and c are three constants, all being determined in order to obtain a Lyapunov function. As A_2 is now invertible choosing $(\alpha, \beta)^\top = A_2^{-\top}\mathbf{r}_2$ i.e. $\alpha = -\frac{1+3\omega_\mathrm{e}}{\varepsilon+3\omega_\mathrm{e}}$ and $\beta = \frac{1}{\varepsilon+3\omega_\mathrm{e}}$, it is easy to check that

$$V'_\varepsilon = x^\alpha y^\beta \left[(b-c)\,\varepsilon x y - (3\omega_\mathrm{e} a + c + 3c\omega_\mathrm{e} + a)\, y - (a+b)\, x\right]$$

Hence, choosing finally $a = -c,\ b = c$ we can construct the function

$$L_{\varepsilon,\omega}(x, y) = \kappa V_{\varepsilon,\omega_\mathrm{e}}(x, y) - 1$$

where

$$V_{\varepsilon,\omega_\mathrm{e}}(x, y) = x^{-\frac{1+3\omega_\mathrm{e}}{\varepsilon+3\omega_\mathrm{e}}} y^{\frac{1}{\varepsilon+3\omega_\mathrm{e}}} (x + y - 1) \text{ and } \kappa^{-1} = V_{\varepsilon,\omega_\mathrm{e}}(\tilde{x}, \tilde{y})$$

one can verify that

1. $L_{\varepsilon,\omega_\mathrm{e}}(\tilde{x}, \tilde{y}) = 0$,
2. $L_{\varepsilon,\omega_\mathrm{e}}(x, y) > 0$ if $(x, y) \in \mathbb{R}^2 \setminus (\tilde{x}, \tilde{y})$
3. $L'_{\varepsilon,\omega_\mathrm{e}}(x, y) = 0$ for all values of $(x, y) \in \mathbb{R}^2$

Hence the function $L(x, y)_{\varepsilon,\omega_\mathrm{e}}$ is a Lyapunov function for this dynamics system and orbits are confined on level curves $L_{\varepsilon,\omega_\mathrm{e}}(x, y) = \mu$ where μ is any positive constant. Such curves are plotted on Fig. 1 for the generic values $\varepsilon = 4$ and $\omega_\mathrm{e} = -1$.

It must be noted that when $\Omega_\mathrm{d}+\Omega_\mathrm{e} < 1$, the dynamics in the dark plane is periodic as all the contour levels of V_ε are closed and all solutions are maximal. The corresponding cosmological solution correspond to endless oscillations of the density parameters $(\Omega_\mathrm{d}, \Omega_\mathrm{e})$ who forever compete with each other for ruling the curvature parameter. Cosmic expansion is in this case an eternal sequence of transient acceleration (when DE dominates) and deceleration (when DM dominates) phases.

Solutions such that $\Omega_\mathrm{d}+\Omega_\mathrm{e} > 1$ are unbounded. They correspond to spatially closed universes, since $\Omega_k < 0$, in which cosmic expansion can reverse into contraction at

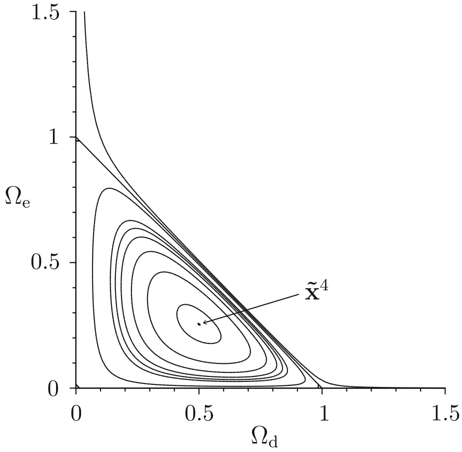

Fig. 1 Contour levels of $L_{4,-1}(x, y)$

some stage, leading to $H = 0$ and consequent singularities in all density parameters. The present formalism with monotonically growing $\lambda = \ln(a)$ cannot extrapolate beyond in vanishing H toward cosmic contraction $H < 0$, since this would imply decreasing λ.

4.3 Attractiveness of the dark plane

We now turn our attention to the behavior of orbits whose initial conditions are not in the dark plane but have non vanishing components in Ω_b and/or Ω_r. Intuitively one could claim that these components are going to vanish because the eigenvalues associated to them are negative, but as the two others, associated to the dark components, are purely imaginary, the equilibrium is no longer a hyperbolic one hence Hartmann–Groβman theorem says that the linear analysis is not sufficient to have a complete description of the system behavior. This point is important and it is often forgotten in the physical literature, it is why we give a counter example in the "Appendix 2". However, even if the invariant manifold methods cannot be straightforwardly used, because the centre manifold is infinitely flat at $\tilde{\mathbf{x}}^4$, we are able, using dynamical systems tools, to prove the attractiveness of the dark plane, for all orbits whose initial conditions belong to the hyper-tetrahedron (8). A detailed proof of the latter statement will be provided in the "Appendix 3".

4.4 Numerical illustration

As we have obtained a general proof of the attractiveness of the dark plane, we give only a simple numerical illustration of this fact. We have numerically solved the dynamical system $(\ln(\mathrm{x}))' = \mathrm{r} + A\mathrm{x}$ with $\mathbf{x} = (\Omega_b, \Omega_d, \Omega_r, \Omega_e)^\top$, the community matrix and the capacity vector defined in (6) with $\varepsilon = 4$. Considering various initial conditions $\mathbf{x}_0$ we always recover an exponential convergence to the dark plane when $\mathbf{x}_0$ has non vanishing first and third components. The Fig. 2 illustrate such a behavior: from the initial condition $\mathbf{x}_0 = (0.3008, 0.2683, 0.0418, 0.2983)^\top$,

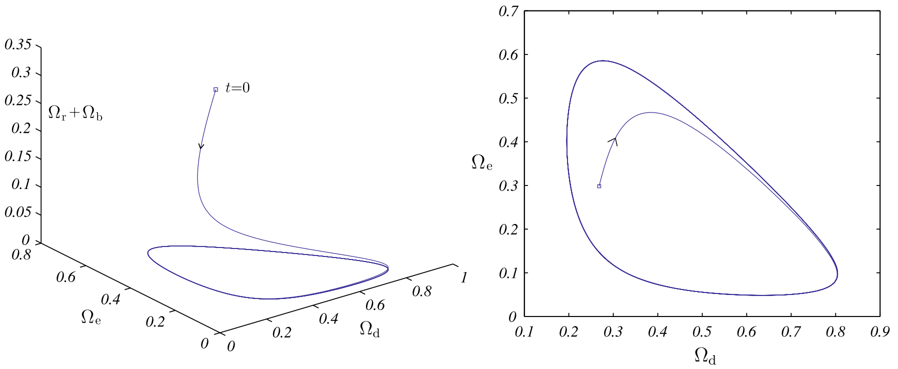

Fig. 2 Time evolution of the orbit inside T_4. *Left panel* 3D plot of the orbit (the *vertical axis* is $\Omega_r(t) + \Omega_b(t)$). *Right panel* 2D projection on the (Ω_d, Ω_e) plane. Parameter and initial conditions: $\varepsilon = 4$, $\Omega_d(0) = 0.2683$, $\Omega_e(0) = 0.2983$, $\Omega_r(0) = 0.0418$ and $\Omega_b(0) = 0.3008$

which belongs to the stable hyper-tetrahedron, we have 3D-plotted the dynamical evolution of the vector $(\Omega_d, \Omega_e, \Omega_b + \Omega_r)^\top$. As expected the third component vanishes and the two others are caught by a contour level of V_4. View from the top in the right part of the Fig. 2 is particularly explicit about this last fact.

5 General correspondence between coupled models and Lotka–Volterra competitive dynamics

In the previous section, we have deduced the general behavior of two coupled species in Jungle Universes. We propose to call such cosmological components twisting species since the special example proposed in the last section represent an eternal exchange between dark energy and dark matter. We will illustrate now that such a behavior can be generalized introducing more couplings.

In this section, we extend the previous discussion to a set of N inter-coupled cosmological species and establish the correspondence with general formulation of competitive Lotka–Volterra models. The goal here is therefore to rewrite the evolution, with the variable $\lambda = \ln(a)$, of cosmological density parameters of interacting fluids under the following Lotka–Volterra form:

$$\mathbf{x}' = \mathrm{diag}(\mathbf{x})\mathbf{f}(\mathbf{x}) \quad \text{with}\, \mathbf{x} \in \mathbb{R}^n \tag{9}$$

where $\mathrm{diag}(\mathbf{x})$ is the diagonal matrix with $\mathbf{x}$ on its diagonal, the ith component of the vector $\mathbf{x}$ denotes the population of the ith species, $\mathbf{f}(\mathbf{x}) = \mathbf{r} + A\mathbf{x}$ is the previously defined linear function which combines the capacity vector $\mathbf{r}$ and the community matrix A. Each coupled fluid characterized by energy density ρ_i, equation of state parameter ω_i and obey the following modified conservation equation:

$$\dot{\rho}_i + 3H\rho_i(1 + \omega_i) = \mathcal{Q}_i;\ i = 1, \ldots, N \tag{10}$$

with the energy balance condition imposing that

$$\sum_{i=1}^{N} \mathcal{Q}_i = 0 \tag{11}$$

where the interaction terms $\mathcal{Q}_i$ take the form of a combination of the involved energy densities:

$$\mathcal{Q}_i = \sum_{j=1}^{N} \beta_{ij} \rho_j . \tag{12}$$

Defining the density parameters $\Omega_i = \frac{8\pi G \rho_i}{3H^2}$, and recalling that the deceleration parameter can be written as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2} \sum_{i=1}^{N} \Omega_i (1 + 3\omega_i)$$

then Eq. (10) becomes

$$\dot{\Omega}_i = \frac{8\pi G \mathcal{Q}_i}{3H^2} + H\Omega_i \left(2 - 3(1 + \omega_i) + \sum_{j=1}^{N} \Omega_j (1 + 3\omega_j) \right) . \tag{13}$$

To rewrite the above equation under Lotka–Volterra form, it is then mandatory to set

$$\mathcal{Q}_i = \sum_{j=1}^{N} \beta_{ij} \rho_j \equiv H\Omega_i \sum_{j=1}^{N} \varepsilon_{ij} \rho_j \tag{14}$$

or, equivalently that the coefficients β_{ij} are no longer constant but are given by

$$\beta_{ij} = H\Omega_i \varepsilon_{ij}$$

with ε_{ij} arbitrary parameters to be specified further. Lotka–Volterra dynamics therefore requires non-linear interaction terms. Given Eq. (14), one can directly rewrite Eq. (13) under Lotka–Volterra form (9) with the following glossary:

$$\begin{aligned} x_i &= \Omega_i \\ (\cdot)' &= \frac{d(\cdot)}{d \ln(a)} \\ r_i &= -(1 + 3\omega_i) \\ A_{ij} &= 1 + 3\omega_j + \varepsilon_{ij} \end{aligned} \tag{15}$$

The energy balance constraint Eq. (11) with the hypothesis (14) now reduces to

$$\sum_{i=1}^{N} \Omega_i \left(\sum_{j=1}^{N} \varepsilon_{ij} \Omega_j \right) = 0 \tag{16}$$

which imposes that the interaction parameters ε_{ij} are *antisymmetric*:

$$\varepsilon_{ij} = -\varepsilon_{ji} \ ; \ \varepsilon_{ii} = 0.$$

In the context of cosmology we find solution of this ODE system in the hyper-tetrahedron

$$\mathcal{T} = \left\{ 1 > \sum_{i=1}^{N} x_1 \right\} \bigcap_{i=1}^{N} \{x_i > 0\}$$

The generalization of the results obtained in the previous section show that ODE system (9) has generically a lot of equilibria but we are interested only by the ones who haven't vanishing component (i.e. the ones not lying on an axis). These "interesting" equilibria are $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}} + \mathbf{r} = \mathbf{0}$. We can now apply this general formulation to the case of several interacting species.

5.1 Two species in interaction

This case $N = 2$ has been treated in details in Sect. 3 for specific values of the equation of state parameters $(\omega_1, \omega_2) = (0, -1)$ and serves here as a validation of the glossary (15). Setting $\varepsilon_{12} = -\varepsilon_{21} \equiv \varepsilon$ the unique non-vanishing component of the interaction tensor ε_{ij}, we obtain after some computation the following equilibria of the cosmological Lotka–Volterra system

$$\Omega_1^{eq} = -\frac{3\omega_2 + 1}{\varepsilon} \tag{17}$$

$$\Omega_2^{eq} = +\frac{3\omega_1 + 1}{\varepsilon} \tag{18}$$

These equilibria are density parameters in open universes ($\Omega_k < 1$) and then must satisfy $0 < \Omega_i^{eq} < 1$. This condition constrains the choice of ε once the choice of the nature of the interacting fluids has been chosen by fixing ω_2 and ω_1.

5.2 The interplay between three coupled species: Jungle triads

Let us set $\varepsilon_{12} = e_1,\ \varepsilon_{13} = e_2$ and $\varepsilon_{23} = e_3$ and compute the corresponding equilibria to find

$$\begin{aligned}\Omega_1^{eq} &= +\frac{e_3 - 3\omega_2 + 3\omega_3}{e_1 - e_2 + e_3}\\ \Omega_2^{eq} &= -\frac{e_2 - 3\omega_1 + 3\omega_3}{e_1 - e_2 + e_3}\\ \Omega_3^{eq} &= +\frac{e_1 - 3\omega_1 + 3\omega_2}{e_1 - e_2 + e_3}\end{aligned} \tag{19}$$

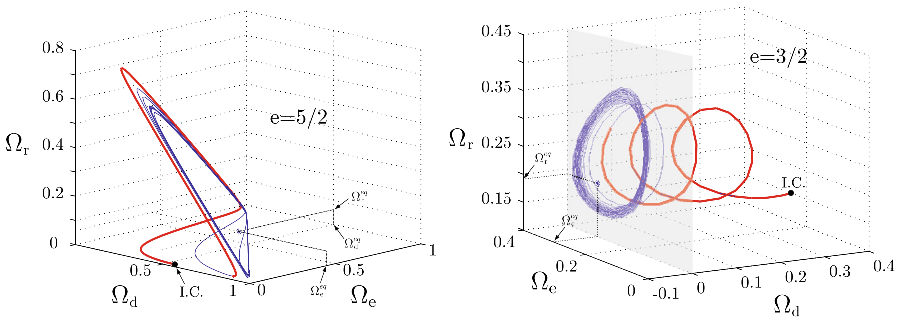

Fig. 3 Evolution of the three coupled density parameters, in the 3D phase space. The beginning of the orbit is overlined. Initial condition is indicated by a *black dot*. Relevant equilibria are indicated by a *star*

Let us remark that in all cases of fluids and coupling we have $\sum_{i=1}^{3} \Omega_i^{eq} = 1$. This fact seems generic for odd values of the number of interacting fluids. If we now impose the fact that the density parameters are comprised between 0 and 1 ($0 < \Omega_i^{eq} < 1$) the constraint on interaction parameters e_1, e_2, e_3 is very complicated, but allows a lot of possibilities. Let us illustrate this case with an example. We consider that the three fluids are made of (1) non-relativistic matter $\omega_1 = 0$, ($x_1 = \Omega_\mathrm{d}$); (2) dark energy $\omega_2 = -1$, ($x_2 = \Omega_\mathrm{e}$) and (3) some relativistic particles $\omega_3 = 1/3$, ($x_3 = \Omega_\mathrm{r}$) all coupled with interaction parameters $e_1 = e_2 = e$ and $e_3 = \varepsilon$. The corresponding equilibria are

$$\Omega_\mathrm{d}^{eq} = \frac{4+\varepsilon}{\varepsilon}, \quad \Omega_\mathrm{e}^{eq} = -\frac{1+e}{\varepsilon} \quad \text{and } \Omega_\mathrm{r}^{eq} = \frac{e-3}{\varepsilon}$$

Providing $\varepsilon < -4$ and $e \in [-1, 3]$ equilibria are cosmologically acceptable. Choosing for example $\varepsilon = -8$, the spectrum of the jacobian matrix near the equilibrium is composed by a real number $\lambda = 1 - \frac{e}{2}$ and two purely imaginary and complex conjugated numbers $\lambda_\pm = \pm\frac{i}{2}\sqrt{2\,|(e+1)(e-3)|}$. When $e \in [-1, 2]$, as $\lambda > 0$ the system twists *outward* $(0, \Omega_\mathrm{e}^{eq}, \Omega_\mathrm{r}^{eq})$ staying in the corresponding 3-tetrahedron, collapsing on the $\Omega_\mathrm{d} = 0$ plane. When $e \in [2, 3]$, as $\lambda < 0$ the system twists *toward* a limit cycle contained in a plane of non vanishing density and including the equilibrium. These results are illustrated on the Fig. 3.

5.3 Jungle quartets

With $N = 4$, the number of free parameters in the scheme (10 in total with 6 for interactions and 4 for equations of state) is too high to be fully constrained by requirements of positiveness and boundedness of density parameters for instance. As for $N = 2$, the positions of the equilibria once again depend on all parameters. If we set $\varepsilon_{12} = e_1$, $\varepsilon_{13} = e_2$, $\varepsilon_{14} = e_3$, $\varepsilon_{23} = e_4$, $\varepsilon_{24} = e_5$ and $\varepsilon_{34} = e_6$, we find that the positions of the equilibria are given by

$$
\begin{aligned}
\Omega_1^{eq} &= -\frac{e_4 - e_5 + e_6 + 3(e_4\omega_4 - e_5\omega_3 + e_6\omega_2)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_2^{eq} &= +\frac{e_2 - e_3 + e_6 + 3(e_2\omega_4 - \omega_3e_3 + e_6\omega_1)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_3^{eq} &= -\frac{e_1 - e_3 + e_5 + 3(e_1\omega_4 + \omega_1e_5 - \omega_2e_3)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_4^{eq} &= +\frac{e_1 - e_2 + e_4 + 3(e_1\omega_3 - e_2\omega_2 + e_4\omega_1)}{e_1e_6 - e_2e_5 + e_4e_3}
\end{aligned}
\tag{20}
$$

Since this system of 4 cosmological coupled species is equivalent to 4D Lotka–Volterra system, chaos can emerge [32] for specific choices of parameters in a so-called normal system where all r_i are positive, which means among cosmological fluids with $\omega_i < -1/3$. As an illustration we propose a double twist in a universe filled by two kinds of dark energy and two kinds of dark matter all interacting. We choose $\omega_1 = -1$, $(x_1 = \Omega_{\mathrm{e},1})$; $\omega_2 = 0$, $(x_2 = \Omega_{\mathrm{d},1})$; $\omega_3 = 0$, $(x_3 = \Omega_{\mathrm{d},2})$ and $\omega_4 = -1$, $(x_4 = \Omega_{\mathrm{e},2})$ for the fluid components, and $e_1 = -4$, $e_2 = 1$, $e_3 = -2$, $e_4 = -1/2$, $e_5 = 1$ and $e_6 = \varepsilon$, we get the following equilibria

$$
\Omega_{\mathrm{e},1}^{eq} = \frac{1}{4},\ \Omega_{\mathrm{d},1}^{eq} = \frac{1}{2},\ \Omega_{\mathrm{d},2}^{eq} = \frac{2}{\varepsilon},\ \Omega_{\mathrm{e},2}^{eq} = \frac{1}{\varepsilon}
$$

The condition on the density parameters then gives $\varepsilon > 12$. Taking $\varepsilon = 16$ we get four complicated but, purely imaginary and conjugated eigenvalues for the Jacobian matrix around the equilibrium:

$$
\lambda_1^{\pm} = \pm i\,\frac{\sqrt{51134 + 6\sqrt{69956601}}}{262} \quad \text{and} \quad \lambda_2^{\pm} = \pm i\,\frac{\sqrt{51134 - 6\sqrt{69956601}}}{262}
$$

The corresponding dynamics is the double twist plotted on Fig. 4.

In the right top panel of Fig. 4, we can have a flavor about the various Ω's dependances in the variable $\lambda = \log a$. In this example the period for the cycles occuring in the dynamics is of order $\lambda = 6$ or 7, it corresponds to an interval of time during which the universes expands by a factor $10^6 - 10^7$: if this interval does not include an inflation period, it is a very long duration in comparison of H_0^{-1}. In other examples one will find other intervals depending on the values of ϵ_{ij} and ω_i.

6 Conclusion

Let us summarize the main points obtained in this paper:

- We have formulated the classical dynamics of Friedmann Universes in the context of the generalized Lotka–Volterra equation. Without coupling, this formulation allows a very simple and pedagogic interpretation of the evolution of these universes. Varying parameters describing the nature of the fluids one can easily understand the corresponding behavior of the so-called Jungle universes.

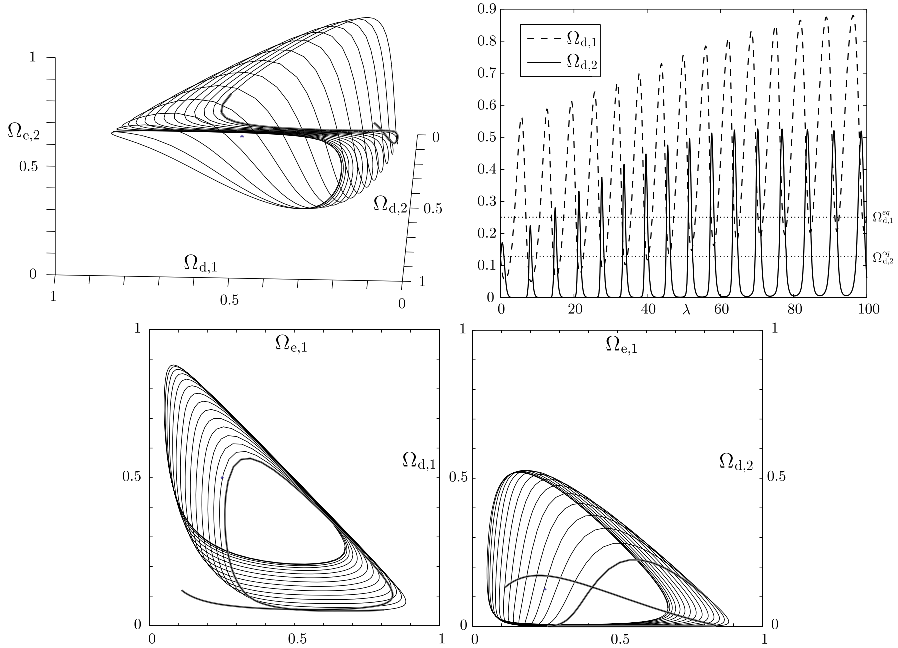

Fig. 4 Jungle quartet: The *left top panel* is a 3D section of the 4D phase space. The *right top panel* is a representation of $\Omega^{eq}_{d,1}(\lambda)$ and $\Omega^{eq}_{d,2}(\lambda)$, the corresponding equilibria are indicated by *doted horizontal lines*. The *two bottom panels* are 2D sections of the 4D phase space. For the phase space sections, the beginning of the orbits are overlined and the relevant equilibria are indicated by a *star*. Initial conditions for the numerical integration are $x_1(0) = 0.11,\ x_2(0) = 0.12,\ x_3(0) = 0.13$ and $x_4(0) = 0.14$

- Cyclic behaviors has been speculated by Lip [16] and Arevalo et al. [17] when FL universe contains exotic “phantom dark matter” fluids (with barotropic index $\omega < -1$) coupled with dark matter; using the generalized Lotka–Volterra formulation of the coupled FL universe we have obtained a general Lyapunov function in the context of the standart cosmological model. This function allows us to rigorously prove the existence of cyclic behavior of FL universe when standart fluids (with barotropic index $\omega > -\frac{1}{3}$) are coupled to dark energy (with barotropic index $\omega < -\frac{1}{3}$).
- In the case of 3 or 4 interacting batrotropic fluids, we have found more complex cyclic behavior of the universe: an expanding twist for $N = 3$ and a double twist for $N = 4$.
- Following the results of the population dynamics, we conjecture that chaos could occur in the dynamics of universes filled by more than 3 interacting fluids: for example, in the case of competitive population dynamics ($r_i < 0$ and $A_{ij} < 0$ in Eq. 15) chaos is the rule (see [32]).

We conclude by claiming that the presented analogy with Lotka–Volterra dynamical systems has offered new unexpected and interesting applications to coupled models in cosmology. Twisting species naturally produce transient phenomena in cosmic expansion, an original feature that could make cosmic coincidence a non unique and therefore less problematic feature.

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Appendix 1: The Lotka–Volterra equation

The Lotka–Volterra equation is defined by a set of two coupled ordinary differential equation of the form

$$\begin{cases} x_1' = x_1 \left(+r_1 - a_{12}x_2\right) \\ x_2' = x_2 \left(-r_2 + a_{21}x_1\right) \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are two function of the variable t, a prime indicate the t-derivative and r_1, r_2, a_{12} and a_{21} are four *positive* constants. It is one of the simplest model to describe interactive dynamical population between some prey (which number is x_1) and its predator (which number is x_2). The constant have intuitive interpretation:

- r_1 (resp. r_2) is the growth rate for preys (resp. predators) to increase (resp. decrease) when there is no predators (resp. preys);
- a_{12} and a_{21} are the coupling factors between the two populations; in ecological systems they are related to the mobility, the agressivity and other natural properties of the different concerned populations.

The properties of such a set of equation are very well known and a lot of classical books (see [24] and reference within) explain and demonstrate that starting with strictly positive initial conditions, the solution $x_1(t)$ and $x_2(t)$ are periodic functions. The orbits in the phase space are concentric closed curves centered on the equilibrium $(\tilde{x}_1, \tilde{x}_2) = \left(\frac{r_1}{a_{12}}, \frac{r_2}{a_{21}}\right)$. These curves correspond to the isocontours of the Ljapunov function $V(x_1, x_2) = a_{21}x + a_{12}y - (r_2 \ln x + r_1 \ln y)$.

When r_1, r_2, a_{12} and a_{21} have different signs the situation is changed and the system can describe other situations like competition, symbiose, etc (see [24]).

Lotka–Volterra systems is often considered as an unrealistic model in the context of population dynamics because there is no limitation when one specie is ruled out. A better model could be implemented introducing a logistic limitation such that

$$\begin{cases} x_1' = x_1 \left(+r_1 - a_{11}x_1 - a_{12}x_2\right) \\ x_2' = x_2 \left(-r_2 + a_{21}x_1 - a_{22}x_2\right) \end{cases}$$

The sign of the new parameters a_{11} and a_{22} is always related to the physical nature of the dynamical system. When all constants are positives, the system admits a general treatment: the Lotka–Volterra curves are generally replaced by spirals which converge to the equilibrium $(x^* \neq 0, y^* \neq 0)$ solution of the system

$$\begin{cases} r_1 = +a_{11}x_1 + a_{12}x_2 \\ r_2 = -a_{21}x_1 + a_{22}x_2 \end{cases}$$

When the matrix $A = \left(a_{ij}\right)$ is non invertible (singular) the situation is called degenerated and equilibria lie on the axes.

Such models could be generalized to n-dimensional systems introducing the generalized Lotka–Volterra equation occuring in this paper. Less things can be said about dynamical properties of such systems when the coefficient of $\mathbf{r}$ and A are generic ones, in particular the dynamics can change strongly giving rise to cyclic competition or cooperation [33].

Appendix 2: Counter example: a linear center which is actually nonlinearly unstable

Consider the dynamical system

$$\begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases}$$

The origin is an equilibrium, the eigenvalues of the jacobian near this equilibrium are $\pm i$. In the linear approximation the origin seems to be a center. But as the eigenvalues have no real part, the system is not hyperbolic and almost nothing can be assumed about the non linear dynamics considering only the linear one around the equilibrium. In this particular case one can prove that the origin is a repulsive focus, and it is then actually unstable. As a matter of fact, considering the intersection M between an orbit and the circle $x^2 + y^2 = R^2$. The angle α between the tangent in M to the orbit and the tangent in M to the circle is given for any radius R by the expression

$$\cos\alpha = (\dot{x}, \dot{y}) \cdot (2x, 2y) = \left(-y + x^3, x + y^3\right) \cdot (2x, 2y) = 2x^4 + 2y^4 > 0$$

Hence, each orbit is going out from any circle of radius $R > 0$ and the origin is unstable.

Appendix 3: Proof of the stability of the $\mathbf{\Omega_d - \Omega_e}$ plane

The aim of this section is to provide a simple proof of the attractiveness of the $\Omega_d - \Omega_e$ plane for all orbits whose initial conditions belong to the hyper-tetrahedron:

$$T_4 = \{\Omega_d > 0\} \cup \{\Omega_e > 0\} \cup \{\Omega_r > 0\} \cup \{\Omega_b > 0\} \cup \{\Omega_d + \Omega_e + \Omega_r + \Omega_b < 1\}\,.$$

Let us recall the ODE system describing the equations of motion:

$$\begin{cases} \Omega_d' = \Omega_d\left[\Omega_d + (\varepsilon + 1 + 3\omega_e)\Omega_e + 2\Omega_r + \Omega_b - 1\right] \\ \Omega_e' = \Omega_e\left[(1 - \varepsilon)\Omega_d + (1 + 3\omega_e)\,\Omega_e + 2\Omega_r + \Omega_b - 1 - 3\omega_e\right] \\ \Omega_r' = \Omega_r\left[\Omega_d + (1 + 3\omega_e)\,\Omega_e + 2\Omega_r + \Omega_b - 2\right] \\ \Omega_b' = \Omega_b\left[\Omega_d + (1 + 3\omega_e)\,\Omega_e + 2\Omega_r + \Omega_b - 1\right] \end{cases} \tag{21}$$

To prove our claim we need to prove first the invariance with respect to the flow (21) of the hyper-tetrahedron T_4. The invariance of each coordinates hyperplanes is trivial and follows straightforwardly from (21). For instance any solution such that $\Omega_d(0) = 0$ will have $\Omega_d(\lambda) = 0$ for all λ, then using the uniqueness of the Cauchy problem we can ensure that any solution with $\Omega_d(0) > 0$ will never cross the hyperplane $\Omega_d = 0$. A very similar analysis can be performed for the remaining cases.

Let us now consider the remaining piece of the boundary of T_4, that is the hyperplane$\{\Omega_d + \Omega_e + \Omega_r + \Omega_b = 1\}$. A straightforward computation gives:

$$(\Omega_d+\Omega_e+\Omega_r+\Omega_b)' = [\Omega_d+(1+3\omega_e)\Omega_e+2\Omega_r+\Omega_b]\,[\Omega_d+\Omega_e+\Omega_r+\Omega_b-1]\,,$$

thus any solution with initial conditions

$$\Omega_d(0) + \Omega_e(0) + \Omega_r(0) + \Omega_b(0) = 1,$$

will always satisfies the constraint

$$\Omega_d(\lambda) + \Omega_e(\lambda) + \Omega_r(\lambda) + \Omega_b(\lambda) = 1 \quad \forall\lambda.$$

Thus once again the uniqueness result of the Cauchy problem implies that any solution such that $\Omega_d(0) + \Omega_e(0) + \Omega_r(0) + \Omega_b(0) < 1$, will never reach the hyperplane $\Omega_d + \Omega_e + \Omega_r + \Omega_b = 1$.

Finally putting together the above partial results, we can conclude that any orbit with initial condition inside T_4 will never leave it.

A by-product of the invariance of the tetrahedron is that orbits inside T_4 will always have positive projections on the axes. This allows us to compute the distance from the plane (Ω_d, Ω_e) using the linear function $F(\Omega_r, \Omega_b) = \Omega_r + \Omega_b$, which is zero if and only if $\Omega_r = \Omega_b = 0$, that is the point belongs to the plane (Ω_d, Ω_e).

We can then compute the Lie derivative of F and prove that its restriction to T_4 is strictly negative, hence $F(\Omega_r(\lambda), \Omega_b(\lambda)) \to 0$ for $\lambda \to +\infty$ and because of the positiveness of $\Omega_r(\lambda)$ and $\Omega_b(\lambda)$ we can conclude that both $\Omega_r(\lambda)$ and $\Omega_b(\lambda)$ goes asymptotically to zero.

To prove the latter claim let us compute the derivative of F along the flow of (21):

$$\left.\frac{dF}{dt}\right|_{\text{flow}} = [\Omega_d + (1+3\omega_e)\Omega_e + 2\Omega_r + \Omega_b - 1]\,[\Omega_r + \Omega_b] - \Omega_r,$$

because of our previous result $\Omega_d(\lambda) + \Omega_b(\lambda) - 1 < -\Omega_e(\lambda) - \Omega_r(\lambda)$ for all λ. Using this inequality we get

$$\left.\frac{dF}{dt}\right|_{\text{flow}} < (3\omega_e\Omega_e+\Omega_r)\,(\Omega_r+\Omega_b) - \Omega_r \; = 3\omega_e\Omega_e\,(\Omega_r+\Omega_b)+(\Omega_r+\Omega_b-1\,)\,\Omega_r$$

let us observe that the right hand side is strictly negative, in fact

$$3\omega_e\Omega_e(\Omega_r + \Omega_b) < 0$$

provided and Ω_e is associated to the dark energy $\left(\omega_e < -\frac{1}{3} < 0\right)$ and

$$\Omega_r + \Omega_b - 1 < \Omega_d + \Omega_e + \Omega_r + \Omega_b - 1 < 0.$$

This concludes our proof of the attractiveness of the dark plane for all orbits whose initial conditions belong to T_4.

References

1. Wainwright, J., Ellis, G.F.R. (eds.): Dynamical Systems in Cosmology. Cambridge University Press. Cambridge, MA (1997)
2. Belinskii, V.A., Khalatnikov, L.M., Lifshitz, E.M.: Oscillatory approach to a singular point in relativistic cosmology. Adv. Phys. **19**, 525 (1970)
3. Misner, C.W.: Mixmaster universe. Phys. Rev. Lett. **22**, 1071 (1969)
4. Wainwright, J., Hsu, L.: A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A. Class. Quantum Gravity **6**, 1409 (1989)
5. Ringström, H.: The Bianchi IX attractor. Ann. Inst. H. Poincaré **2**, 405 (2001)
6. Damour, T., Henneaux, M., Nicolai, H.: Cosmological billiards. Class. Quantum Gravity **20**, R145 (2003)
7. Caldwell, R.R., Kamionkowski, M., Weinberg, N.N.: Phantom energy and cosmic doomsday. Phys. Rev. Let. **91**, 7 (2003)
8. Amendola, L., Tsujikawa, S.: Dark Energy: Theory and Observations. Cambridge University Press, Cambridge, MA (2010)
9. Copeland, E.J., Liddle, A., Wands, D.: Exponential potentials and cosmological scaling solutions. Phys. Rev. D **57**, 4686–4690 (1998)
10. de la Macorra, A., Piccinelli, G.: General scalar fields as quintessence. Phys. Rev. D **61**, 123503 (2000)
11. Ng, S.C.C., Nunes, N.J., Rosati, F.: Applications of scalar attractor solutions to cosmology. Phys. Rev. D **64**, 083510 (2001)
12. Chimento, L.P., Jakubi, A.S., Pavon, D.: Enlarged quintessence cosmology. Phys. Rev. D **62**, 063508 (2000)
13. Copeland, E.J., Sami, M., Tsujikawa, S.: Dynamics of dark energy. Int. J. Mod. Phys. D **15**, 1753 (2006). arXiv:hep-th/0603057
14. Tsujikawa, S.: Dark energy investigation and modeling. In: Dark Matter and Dark Energy Astrophysics and Space Science Library, vol. 370, pp. 331–402 (2011) arXiv:1004.1493
15. Kremer, G.M., Sobreiro, O.A.S: Bulk viscous cosmological model with interacting dark fluids, Braz. J. Phys. **42**, 77 (2012) arXiv:1109.5068 [gr-qc]
16. Lip, S.Z.W.: Interacting cosmological fluid and the coincidence problem. Phys. Rev. D **83**, 023528 (2011)
17. Arévalo, F., Bacalhau, A.P., Zimdahl, W.: Cosmological dynamics with non-linear interactions. Class. Quantum Gravity **29**, 235001 (2012)
18. Zimdahl, W., Pavon, D.: Scaling cosmology. Gen. Relativ. Gravit **35**, 413 (2003)
19. Lotka, A.J.: Contribution to the theory of periodc reaction. J. Phys. Chem **14**(3), 271–274 (1910)
20. Volterra, V.: Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. Mem. Acad. Lincei Roma **2**, 31–113 (1926)
21. Goel, N.S., et al.: On the Volterra and Other Non-Linear Models of Interacting Populations. Academic Press, New York (1971)
22. Uzan, J.-P., Lehoucq, R.: A dynamical study of the Friedmann equations. Eur. J. Phys. **22**, 371–384 (2001)
23. Hobson, M.P., Efstathiou, G.P., Lasenby, A.N.: General Relativity: An Introduction for Physicists. Cambridge University Press, Cambridge, MA (2006)
24. Hofbauer, J., Sigmund, K.: Evolutionary Games and Population Dynamics. Cambridge University Press, Cambridge, MA (1998)
25. Murray, J.D.: Mathematical Biology I: An Introduction, 3rd edn. Springer, Berlin (2002)
26. Martin, J.: Inflationary cosmological perturbations of quantum-mechanical origin. In: Lecture Notes Physics, vol. 669, pp. 199–244 (2005)

27. Caldera-Cabra, G., Maartens, R.: Arturo Urena-Lopez, L.: Dynamics of interacting dark energy. Phys. Rev. D **79**, 063518 (2009)
28. Wetterich, C.: An asymptotically vanishing time-dependent cosmological "constant". Astron. Astrophys. **301**, 321 (1995)
29. Amendola, L.: Coupled quintessence. Phys. Rev. D **62**, 043511 (2000)
30. Cen, R.: Decaying cold dark matter model and small-scale power. Astrophys. J. Lett **546**(L77), 2001 (2000)
31. Uzan, J.-P.: Dynamics of relativistic interacting gases: from a kinetic to a fluid description. Class. Quantum Gravity **15**, 1063 (1998)
32. Wang, R., Xiao, D.: Bifurcations and chaotic dynamics in a 4-dimensional competitive Lotka–Volterra system. Nonlinear Dyn. **59**, 411–422 (2010)
33. May, R.M., Leonard, W.J.: Non linear aspects of competition between three species. SIAM J. Appl. Math. **29**(2), 243–253 (1975)